

# On reflection from a half-space with negative real permittivity and permeability: Time-domain and frequency-domain results

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**ABSTRACT:** The reflection of a normally incident wideband pulse by a half-space whose permittivity and permeability obey the one-resonance Lorentz model is calculated. The results are compared with those from frequency-domain reflection analysis. In the spectral regime wherein the real parts of the permittivity and the permeability are negative, as well as in small adjacent neighborhoods of that regime, the time-domain analysis validates the conclusion that it is sufficient to ensure that the imaginary part of the refractive index is positive — regardless of the sign of the real part of the refractive index.

**Key words:** *negative real permeability; negative real permittivity; time-domain analysis*

## 1 Introduction

Take a half-space occupied by an isotropic, homogeneous, dielectric medium of relative permittivity  $\epsilon_r(\lambda_0) = \epsilon'_r(\lambda_0) + i\epsilon''_r(\lambda_0)$ , where  $\lambda_0$  is the free-space wavelength.

When a plane wave with electric field phasor of unit amplitude is incident normally on this half-space, the amplitude  $r(\lambda_0)$  of the electric field phasor of the reflected plane wave is given by

$$\left. \begin{aligned} r(\lambda_0) &= \frac{n(\lambda_0)-1}{n(\lambda_0)+1} \\ n(\lambda_0) &= \sqrt{\epsilon_r(\lambda_0)} \end{aligned} \right\}, \quad (1)$$

where  $n(\lambda_0)$  is the refractive index. The foregoing standard result is found in countless many textbooks; see, e.g., Born & Wolf [1] and Jackson [2]. It can be applied to insulators, poor conductors as well as good conductors.

Let the medium occupying the half-space also have magnetic properties represented by the relative permeability  $\mu_r(\lambda_0) = \mu'_r(\lambda_0) + i\mu''_r(\lambda_0)$ . Then, the previous result has to be modified as follows:

$$\left. \begin{aligned} r(\lambda_0) &= \frac{[n(\lambda_0)/\mu_r(\lambda_0)]-1}{[n(\lambda_0)/\mu_r(\lambda_0)]+1} \\ n(\lambda_0) &= \sqrt{\epsilon_r(\lambda_0) \mu_r(\lambda_0)} \end{aligned} \right\}. \quad (2)$$

This result (2) for the reflection coefficient  $r(\lambda_0)$  gains particular prominence in the context of composite materials that supposedly exhibit  $\{\epsilon'_r < 0, \mu'_r < 0\}$  and  $\{\epsilon''_r \geq 0, \mu''_r \geq 0\}$  in some spectral regime [3] — the time-dependence  $\exp(-i\omega t)$  being implicit for all electromagnetic fields, with  $\omega$  as the angular frequency. Shelby *et al.* [3] argued that the real part of  $n(\lambda_0) = n'(\lambda_0) + in''(\lambda_0)$  must then be chosen to be negative. Consequently, it was concluded [5] that “[p]lane waves appear to propagate from plus and minus infinity towards the source ..... [but] ..... clearly energy propagates outwards from the source.” The conclusion has been employed to deduce the amplification of evanescent plane waves, leading to a proposal for aberration-free lenses [6].

Certainly, energy does not flow in the *backward* direction. Rather, the forward direction is defined as the direction of the energy transport, while the phase velocity

*may* then be pointed in the backward direction — a known phenomenon [7]. Assuming the one-resonance Drude model for both  $\epsilon_r(\lambda_0)$  and  $\mu_r(\lambda_0)$ , confining themselves to the spectral regime wherein  $\epsilon'_r(\lambda_0) < 0$  and  $\mu'_r(\lambda_0) < 0$ , and impedance-matching the dielectric-magnetic half-space to free space, Ziolkowski and Heyman [4] pointed out that the reflection coefficient  $r(\lambda_0)$  must have finite magnitude. Employing the one-resonance Lorentz model for both  $\epsilon_r(\lambda_0)$  and  $\mu_r(\lambda_0)$ , McCall *et al.* [8] recently deduced that

- (i) both  $\epsilon'_r(\lambda_0)$  and  $\mu'_r(\lambda_0)$  need not be negative for  $n'(\lambda_0)$  to be negative and  $n''(\lambda_0)$  to be positive, and
- (ii) the magnitude of the reflection coefficient  $r(\lambda_0)$  does not then exceed unity.

Let us emphasize that the spectral regime for  $n'(\lambda_0) < 0$  overflows into the neighborhoods of the regime wherein both  $\epsilon'_r(\lambda_0)$  and  $\mu'_r(\lambda_0)$  are negative.

Whereas frequency-domain analysis for the response of a half-space calls for the determination of the signs of  $n'(\lambda_0)$  and  $n''(\lambda_0)$ , the refractive index does not enter time-domain analysis. Therefore, time-domain analysis offers an independent way of assessing the results of frequency-domain analysis. In this communication, we present our conclusions from the response of a dielectric-magnetic half-space with negative real refractive index in a certain spectral regime to a normally incident wideband pulse. The Fourier transform of the reflected pulse confirms the frequency-domain calculations of  $r(\lambda_0)$  in that spectral regime, and underscores the requirement of  $n''(\lambda_0) > 0$  for real materials (which cannot be non-dissipative due to causality) to be sufficient.

## 2 Theory in Brief

Consider the region  $\{z \geq 0, t \geq 0\}$  of spatiotemporal space. The half-space  $z \geq z_\ell$  is occupied by a homogeneous dielectric-magnetic medium whose constitutive relations are given by

$$\left. \begin{aligned} \mathbf{D}(\mathbf{r}, t) &= \epsilon_0 [\mathbf{E}(\mathbf{r}, t) + (\chi_e * \mathbf{E})(\mathbf{r}, t)] \\ \mathbf{B}(\mathbf{r}, t) &= \mu_0 [\mathbf{H}(\mathbf{r}, t) + (\chi_m * \mathbf{H})(\mathbf{r}, t)] \end{aligned} \right\}, \quad z \geq z_\ell, \quad (3)$$

where the asterisk denotes the convolution operation [9] with respect to time, while the susceptibility functions

$$\begin{aligned} \chi_{e,m}(t) &= p_{e,m} \frac{2\pi c_0}{\lambda_{e,m}} \sin\left(\frac{2\pi c_0 t}{\lambda_{e,m}}\right) \\ &\quad \times \exp\left(-\frac{c_0 t}{M_{e,m} \lambda_{e,m}}\right) \mathcal{U}(t) \end{aligned} \quad (4)$$

obey the one-resonance Lorentz model. These susceptibility functions correspond to

$$\epsilon_r(\lambda_0) = 1 + \frac{p_e}{1 + (N_e^{-1} - i\lambda_e \lambda_0^{-1})^2}, \quad (5)$$

$$\mu_r(\lambda_0) = 1 + \frac{p_m}{1 + (N_m^{-1} - i\lambda_m \lambda_0^{-1})^2} \quad (6)$$

in the frequency domain [10]. In these expressions,  $\epsilon_0$  and  $\mu_0$  are the permittivity and the permeability of free space;  $c_0 = (\epsilon_0 \mu_0)^{-1/2}$  is the speed of light in free space;  $\mathcal{U}(t)$  is the unit step function;  $p_{e,m}$  denote the so-called oscillator strengths; while  $N_{e,m} = 2\pi M_{e,m}$  and  $\lambda_{e,m}$  determine the resonance wavelengths and the linewidths. The region  $z \leq z_\ell$  is vacuum.

At time  $t = 0$ , an amplitude-modulated wave is supposedly launched normally from the plane  $z = 0$ ; therefore, all fields are independent of  $x$  and  $y$ . The initial

and boundary conditions on  $\mathbf{E}(z, t)$  and  $\mathbf{H}(z, t)$  are as follows:

$$\left. \begin{aligned} \mathbf{E}(z, 0) &= \mathbf{0} \quad \forall z \geq 0 \\ \mathbf{H}(z, 0) &= \mathbf{0} \quad \forall z \geq 0 \\ \mathbf{E}(0, t) &= g(t) \sin\left(\frac{2\pi c_0 t}{\lambda_{car}}\right) \hat{\mathbf{y}} \quad \forall t \geq 0 \\ \mathbf{H}(0, t) &= -(\epsilon_0/\mu_0)^{1/2} g(t) \sin\left(\frac{2\pi c_0 t}{\lambda_{car}}\right) \hat{\mathbf{x}} \quad \forall t \geq 0 \end{aligned} \right\}. \quad (7)$$

Whereas  $\lambda_{car}$  is the carrier wavelength, the function

$$g(t) = \frac{c_0 t}{2\lambda_{car}} \exp(-c_0 t/\lambda_{car}) \quad (8)$$

was chosen to represent the pulse. The cartesian unit vectors are denoted by  $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ .

The differential equations to be solved are as follows:

$$\frac{\partial}{\partial z} E_y(z, t) = \mu_0 \left[ \frac{\partial}{\partial t} H_x(z, t) + \frac{\partial}{\partial t} (\chi_m * H_x)(z, t) \right], \quad (9)$$

$$\frac{\partial}{\partial z} H_x(z, t) = \epsilon_0 \left[ \frac{\partial}{\partial t} E_y(z, t) + \frac{\partial}{\partial t} (\chi_e * E_y)(z, t) \right]. \quad (10)$$

Their solution was carried out using a finite difference calculus described elsewhere in detail [11]. It suffices to state here that  $z$  and  $t$  were discretized into segments of size  $\Delta z$  and  $\Delta t$ , respectively; derivatives were replaced by central differences, and the leapfrog method was used [12].

Finally, the Fourier transform

$$\tilde{E}_y(z, \lambda_0) = \int_{t_a}^{t_b} E_y(z, t) \exp(-i \frac{2\pi c_0}{\lambda_0} t) dt \quad (11)$$

was calculated to determine the spectral contents of the incident and the reflected pulses. The parameters  $z$ ,  $t_a$  and  $t_b$  were chosen to capture as much of both pulses as possible, following the procedure described elsewhere [11]. The computed ratio of the Fourier transform of the reflected pulse to that of the incident pulse is denoted here by  $r_{TD}(\lambda_0)$ , the subscript TD indicating its emergence from time-domain analysis.

### 3 Numerical Results and Discussion

For the sake of illustration, the following values were selected for the constitutive parameters of the dielectric-magnetic half-space:  $p_e = 1$ ,  $p_m = 0.8$ ,  $\lambda_e = 300$  nm,  $\lambda_m = 320$  nm,  $M_e = M_m = 100$ . Thus,  $\epsilon_r''(\lambda_0) > 0$  and  $\mu_r''(\lambda_0) > 0$  for all  $\lambda_0$ . However,  $\epsilon_r'(\lambda_0)$  is negative for  $\lambda_0 \in [212.1, 300]$  nm, but it is positive for all other  $\lambda_0$ ; while  $\mu_r'(\lambda_0)$  is negative for  $\lambda_0 \in [238.6, 320]$  nm, and positive for all other  $\lambda_0$ .

The definition of the refractive index in (2) suggests two possibilities: Either

- A.  $n''(\lambda_0)$  is negative for  $\lambda_0 \in [236.1, 316.8]$  nm and positive elsewhere, consistent with the requirement of  $n'(\lambda_0) > 0 \forall \lambda_0$ ; or
- B.  $n'(\lambda_0)$  is negative for  $\lambda_0 \in [236.1, 316.8]$  nm and positive elsewhere, consistent with the requirement of  $n''(\lambda_0) \geq 0 \forall \lambda_0$ .

Thus, our attention had to be focussed on the anomalous spectral regime  $\lambda_0 \in [236.1, 316.8]$  nm. In this regime, the reflection coefficient  $r(\lambda_0)$  for Possibility A is the reciprocal of that for Possibility B. The two possibilities can therefore be unambiguously distinguished from one another.

The carrier wavelength was chosen as  $\lambda_{car} = 240$  nm. The pulse duration is 3 fs and its 3dB band is  $\lambda_0 \in [218, 261]$  nm. Therefore the anomalous spectral regime was substantively covered by our time-domain calculations. The segment sizes  $\Delta z = 5$  nm and  $\Delta t = 0.015$  fs used by us were adequate for the chosen constitutive parameters, but obviously would be totally inadequate in the resonance bands of  $\epsilon_r(\lambda_0)$  and  $\mu_r(\lambda_0)$ .

Possibility A is clearly nonsensical. It implies transport of energy in the half-space  $z \geq z_\ell$  towards the interface  $z = z_\ell$ . Not surprisingly therefore, (2), (5) and (6) yielded  $|r(\lambda_0)| > 1$  for all  $\lambda_0 \in [236.1, 316.8]$  nm.

Figure 1 presents the computed values of  $|r(\lambda_0)|$  obtained from (2), (5) and (6) for Possibility B (i.e., when  $n''(\lambda_0) \geq 0$  is guaranteed for all  $\lambda_0$ ). The computed values of  $|r_{TD}(\lambda_0)|$  are also shown therein. The two sets of magnitudes compare very well for  $\lambda_0 \leq 290$  nm. Examination of the refracted pulse also showed that it transported energy away from the interface  $z = z_\ell$ , which corroborates the observations of Ziolkowski and Heyman [4].

Thus, time-domain analysis validates the conclusion that  $n(\lambda_0)$  must be selected in frequency-domain research in such a way that  $n''(\lambda_0) \geq 0$  — irrespective of the sign of  $n'(\lambda_0)$ .

## References

- [1] M. Born and E. Wolf, Principles of Optics, Pergamon, Oxford, UK, 1987, p. 41.
- [2] J.D. Jackson, Classical Electrodynamics, Wiley, New York, 1999; p. 306.
- [3] R.A. Shelby, D.R. Smith and S. Schultz, Experimental verification of a negative index of refraction, Science 292 (2001), 77–79.
- [4] R.W. Ziolkowski and E. Heyman, Wave propagation in media having negative permittivity and permeability. Phys Rev E 64 (2001), 056625.
- [5] D.R. Smith and N. Kroll, Negative refractive index in left-handed materials, Phys Rev Lett 85 (2000), 2933–2936. [These materials do not possess either structural or intrinsic chirality.]
- [6] J.B. Pendry, Negative refraction makes a perfect lens. Phys Rev Lett 85 (2001), 3966–3969. [See also correspondence on this paper: G.W. 't Hooft, Phys Rev

- Lett 87 (2001) 249701; J. Pendry, Phys Rev Lett 87 (2001) 249702; J.M. Williams, Phys Rev Lett 87 (2001) 249703; J. Pendry, Phys Rev Lett 87 (2001) 249704. In addition, see appraisals by Ziolkowski & Heyman [4] and Lakhtakia (*arXiv:physics/0112004* on <http://xxx.lanl.gov>).]
- [7] See the many instances cited by: I.V. Lindell, S.A. Tretyakov, K.I. Nikoskinen and S. Ilvonen, BW media — Media with negative parameters, capable of supporting backward waves, Microw. Opt Technol Lett 31 (2001), 129–133.
- [8] M.W. McCall, A. Lakhtakia and W.S. Weiglhofer, The negative index of refraction demystified. University of Glasgow, Department of Mathematics Preprint No. 2001/30.
- [9] J.W. Goodman, Introduction to Fourier optics, McGraw–Hill, New York, 1968, p. 19.
- [10] C.F. Bohren and D.R. Huffman, Absorption and scattering of light by small particles, Wiley, New York, 1983, Sec. 9.1.
- [11] J.B. Geddes III and A. Lakhtakia, Time–domain simulation of the circular Bragg phenomenon exhibited by axially excited chiral sculptured thin films, Eur Phys J Appl Phys 14 (2001), 97–105; erratum: 16 (2001), 247.
- [12] N. Gershenfeld, The nature of mathematical modeling, Cambridge Univ. Press, Cambridge, UK, 1999, Sec. 7.1.



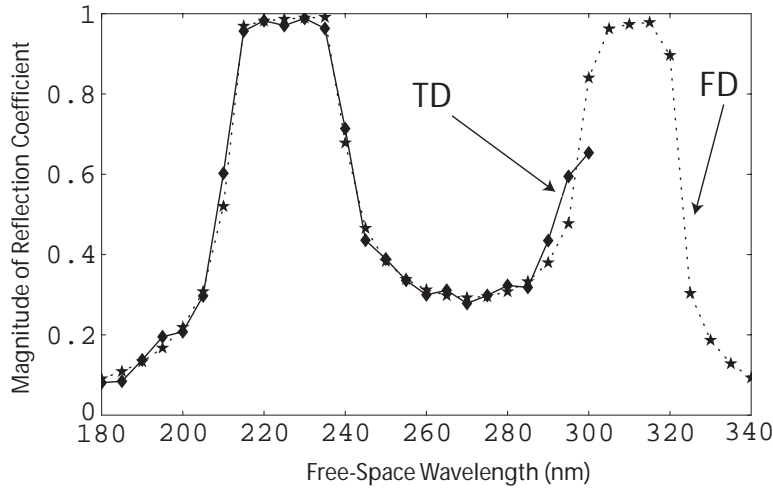


Figure 1: Magnitudes of the reflection coefficient computed using two different methods. The frequency–domain equations (2), (5) and (6) yield the plot labeled FD, after ensuring that  $n''(\lambda_0)$  is positive for all  $\lambda_0$ . The time–domain analysis, followed by the Fourier transformations of the electric field associated with the incident and the reflected pulses, as discussed in Section 2, yields the plot labeled TD. The time–domain analysis fails in the resonance bands of  $\epsilon_r$  and  $\mu_r$  because the chosen discretization of space and time is necessarily very coarse therein.